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SOME FINITELY ADDITIVE VERSIONS OF THE STRONG LAW OF LARGE NUMBERS[†]

BY
ROBERT CHEN

ABSTRACT

Let X be a non-empty set, $H = X^\infty$, $\sigma = \gamma_1 \times \gamma_2 \times \cdots$ be an independent strategy on H, and $\{Y_n\}$ be a sequence of coordinate mappings on H. The following strong law in a finitely additive setting is proved: For some constant $r \geqq 1$, if $\Sigma_{n=1}^\infty \{\sigma(|Y_n|^{2r})/n^{1+r}\} < \infty$ and $\sigma(Y_n) = 0$ for all $n = 1, 2, \cdots$, then $\frac{1}{n}\Sigma_{j=1}^n Y_j$ converges to 0 with σ-measure 1 as $n \to \infty$. The theorem is a generalization of Chung's strong law in a coordinate representation process. Finally, Kolmogorov's strong law in a finitely additive setting is proved by an application of the theorem.

1. Introduction

Let X be a non-empty set with the discrete topology, $H = X^\infty$ with the product topology, and $F(X)$ be the set of all finitely additive probability measures defined on the class of all subsets of X. As defined by Dubins and Savage [4], a strategy σ on H is a sequence $(\sigma_0, \sigma_1, \sigma_2, \cdots)$, where σ_0 is in $F(X)$, and, for each positive integer n, σ_n is a mapping from X^n to $F(X)$. For any positive integer n, any element $(x_1, x_2, \cdots, x_n)$ in X^n is called a partial history with length n. Suppose that σ is a strategy on H, $p = (x_1, x_2, \cdots, x_n)$ is a partial history with length n, then the conditional strategy given the partial history p with respect to the strategy σ is a strategy (written $\sigma[p]$) on H defined by (i) $(\sigma[p])_0 = \sigma_n(p) = \sigma_n(x_1, x_2, \cdots, x_n)$, i.e., $(\sigma[p])_0$ is just the finitely additive probability measure $\sigma_n(x_1, x_2, \cdots, x_n)$, and (ii) for any positive integer m, $(\sigma[p])_m$ is a mapping from X^m to $F(X)$ defined by $(\sigma[p])_m(x'_1, x'_2, \cdots, x'_m) = \sigma_{n+m}(x_1, x_2, \cdots, x_n, x'_1, x'_2, \cdots, x'_m)$ for all $(x'_1, x'_2, \cdots, x'_m)$ in X^m. In [7], Purves and Sudderth call a strategy σ on H independent, if there exists a sequence $\{\gamma_n\}$ in $F(X)$ such that

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$\sigma_0 = \gamma_1$, and for each positive integer n and all n-tuple $p = (x_1, x_2, \cdots, x_n)$ in X^n, $\sigma_n(p) = \gamma_{n+1}$ and they write $\gamma_1 \times \gamma_2 \times \cdots$ for σ. If, in addition, there is a finitely additive probability measure γ in $F(X)$ such that $\gamma = \gamma_1 = \gamma_2 = \cdots$, then σ is said to be independent and identically distributed, and $\gamma \times \gamma \times \cdots$ is written for such a strategy.

In [7], Purves and Sudderth showed that if σ is a strategy on H, then there exists a field $\mathscr{A}(\sigma)$ containing all Borel subsets of H and a finitely additive probability measure (still denoted by σ) such that σ is defined on $\mathscr{A}(\sigma)$ with some nice properties. Based on this result, we can consider $(H, \mathscr{A}(\sigma), \sigma)$ as a finitely additive probability space and a standard theory of integration with respect to the finitely additive probability measure σ on the field $\mathscr{A}(\sigma)$ is available (cf [2], [5]). Later we will use $\sigma(Y)$ to denote the integral of the real-valued function Y on H with respect to the strategy σ.

A sequence $\{Y_n\}$ of real-valued functions on H is called a sequence of coordinate mappings on H if the function Y_n depends only on the nth coordinate for all $n = 1, 2, \cdots$. A sequence $\{Y_n\}$ of real-valued functions on H is called a sequence of identical, coordinate mappings on H if $\{Y_n\}$ is a sequence of coordinate mappings on H which satisfies the following property: For each pair (m, n) of positive integers, $Y_m(h) = Y_n(h)$ whenever $x_m = x_n$ and $h = (x_1, x_2, \cdots, x_m, \cdots, x_n, \cdots)$ is in H.

In [7], Purves and Sudderth have shown that if σ is an independent strategy on H and $\{Y_n\}$ is a sequence of uniformly bounded coordinate mappings on H such that $\sigma(Y_n) = 0$ for all $n = 1, 2, \cdots$, then the set

$$A = \left[h \,\middle|\, \lim_{n \to \infty} 1/n \sum_{j=1}^{n} Y_j(h) = 0 \right]$$

has σ-measure one. In this paper, we show that the above result still holds without the boundedness assumption (see Theorems 4.1, 4.2 below). Furthermore, we also show that Kolmogorov's strong law of large numbers holds for independent, identically distributed strategies and sequences of identical, coordinate mappings (see Theorem 4.3 below).

2. Preliminary definitions and some useful lemmas

Throughout this paper, X is a non-empty set with the discrete topology and $H = X^\infty$ with the product topology. Subsets of H which are simultaneously closed and open in this topology will be referred to as clopen. If K is a subset of H, Y is a real-valued function on H, and $p = (x_1, x_2, \cdots, x_n)$ is a partial history, then the set Kp is defined by

$$Kp = [h' = (x'_1, x'_2, \cdots) \mid ph' = (x_1, x_2, \cdots, x_n, x'_1, x'_2, \cdots) \in K]$$

and the function Yp on H is defined by $Yp(h') = Y(ph')$ for all h' in H.

DEFINITION 2.1. A stop rule τ on H is a mapping from H to the set of all positive integers such that if h, h' are in H and h' agrees with h through the first $\tau(h)$ coordinates, then $\tau(h') = \tau(h)$. An incomplete stop rule τ^* on H is a mapping from H to the set of all positive integers and ∞ such that if h, $h' \in H$ and h' agrees with h throughout the first $\tau^*(h)$ coordinates, then $\tau^*(h') = \tau^*(h)$.

DEFINITION 2.2. A subset K of H is said to be determined by a stop rule τ on H if and only if, "for any h in K and any h' in H, if h' agrees with h through the first $\tau(h)$ coordinates, then h' is in K."

The proof of Lemmas 2.1, 2.2, 2.3 is straightforward and the details are presented in [2].

LEMMA 2.1. *Let $\sigma = \gamma_1 \times \gamma_2 \times \cdots$ be an independent strategy on H, Y be a real-valued function on H which depends only on the nth coordinate, and f be a real-valued function on X such that $Y(h) = f(x_n)$ whenever $h = (x_1, x_2, \cdots, x_n, \cdots)$ in H. Then we have*:

(i) *f is γ_n-integrable if and only if Y is σ-integrable,*

(ii) *Y is σ-integrable if and only if Yp is $\sigma[p]$-integrable and the $\sigma[p]$-integral of Yp is independent of p for all $p = (x_1, x_2, \cdots, x_{n-1})$ in X^{n-1}.*

Furthermore,

$$\gamma_n(f) = \sigma[p](Yp) = \sigma(Y)$$

for all p in X^{n-1} and

$$\sigma(Y) = \int \cdots \int \sigma[p](Yp)d\gamma_{n-1}\, d\gamma_{n-2} \cdots d\gamma_1$$

whenever these integrals exist.

LEMMA 2.2. *Let $\sigma = \gamma_1 \times \gamma_2 \times \cdots$ be an independent strategy on H, $\{Y_n\}$ be a sequence of coordinate mappings, and $1 \leqq r < \infty$. Suppose that $|Y_1|^r, |Y_2|^r, \cdots$ are σ-integrable. Then $|\Sigma_{j=m}^n Y_j|^r$ is σ-integrable for all $1 \leqq m < n < \infty$. Furthermore, $\sigma[p](|\Sigma_{j=m}^n Y_j|^r p) = \sigma(|\Sigma_{j=m}^n Y_j|^r)$ for all p in X^{m-1}.*

LEMMA 2.3. *Let $\sigma = \gamma_1 \times \gamma_2 \times \cdots$ be an independent strategy on H and $\{Y, Z\}$ be two real-valued, σ-integrable functions on H. Suppose that Y depends only on the first l coordinates and Z depends only on those coordinates with indexes from $l + m$ to $l + m + n$ for some positive integers l, m, n. Then YZ is σ-integrable and $\sigma(YZ) = \sigma(Y)\sigma(Z)$.*

LEMMA 2.4. *Let $\sigma = \gamma_1 \times \gamma_2 \times \cdots$ be an independent strategy on H and $\{Y_n\}$ be a sequence of coordinate mappings such that $\sigma(Y_n) = 0$, $\sigma(|Y_n|^{2r}) < \infty$ for all $n = 1, 2, \cdots$, where r is a constant and $r \geqq 1$. Then there is a positive constant A such that*

$$\sigma\left(\left|\sum_{j=m+1}^{n} Y_j\right|^{2r}\right) \leqq A(n-m)^{r-1} \sum_{j=m+1}^{n} \sigma(|Y_j|^{2r})$$

for all $0 \leqq m < n < \infty$ and the constant A depends only on r and not on m or n.

PROOF. The proof of this lemma is essentially the same as the one in the conventional theory of probability (see [6]) and it is too lengthy to present here.

LEMMA 2.5. *Let $\{a_n\}$ be a sequence of non-negative real numbers and $\{b_n\}$ be a non-decreasing sequence of positive real numbers such that $b_n \to \infty$ as $n \to \infty$. If $\Sigma_{n=1}^{\infty} a_n / b_n < \infty$, then there is a sequence $\{c_n\}$ of positive real numbers satisfying:*

(i) $c_n b_{2^n}^{-1} \Sigma_{j=1}^{2n} a_j \to 0$ *as* $n \to \infty$,

(ii) $1 \leqq c_n \leqq c_{n+1} \leqq c_n + 1$ *for all* $n = 0, 1, 2, \cdots$ *and* $c_n \to \infty$ *as* $n \to \infty$,

(iii) $\Sigma_{n=0}^{\infty} c_n \{\Sigma_{2^{n-1} < j \leqq 2^n} a_j / b_j\} < \infty$.

PROOF. For each $n = 1, 2, \cdots$, let $d_n = b_{2^n}^{-1} \Sigma_{j=1}^{2^n} a_j$. By Kronecker's lemma, $d_n \to 0$ as $n \to \infty$. Hence there is a strictly increasing sequence $\{n_j\}$ of positive integers such that $d_n \leqq j^{-2}$ if $n \geqq n_j$, $j = 1, 2, \cdots$. Define $e_n = 1$ if $0 \leqq n < n_1$, $e_n = j$ if $n_j \leqq n < n_{j+1}$, $j = 1, 2, \cdots$. Then $d_n e_n \leqq j^{-1}$ if $n \geqq n_j$, $j = 1, 2, \cdots$ and hence $d_n e_n \to 0$ as $n \to \infty$.

By assumption, we can and do assume that $\Sigma_{n=1}^{\infty} a_n / b_n \leqq 1$. Let $l_0 = 0$ and let

$$l_i = \max\left\{ l_{i-1} + 1, \inf\left\{ j \,\middle|\, \sum_{n=j}^{\infty} \frac{a_n}{b_n} \leqq 2^{-i} \right\}\right\}, \quad i = 1, 2, \cdots.$$

It is easy to see that $1 \leqq l_1 < l_2 < \cdots < \infty$ and $l_n \to \infty$ as $n \to \infty$. Now, for each $i = 1, 2, \cdots$ and each $j = 1, 2, \cdots$, let $u_{ij} = 0$ if $j < l_i$, $u_{ij} = 1$ if $j \geqq l_i$. Then

$$\sum_{j=1}^{\infty} u_{ij} a_j b_j^{-1} = \sum_{j=l_i}^{\infty} a_j b_j^{-1} \leqq 2^{-i}$$

for all $i = 1, 2, \cdots$. Let $v_j = 1$ if $j = 1, 2, \cdots l_1 - 1$ and $v_j = \Sigma_{i=1}^{\infty} u_{ij}$ if $j \geqq l_1$, then it is easy to see that $v_j \geqq i$ if $j \geqq l_i$ for all $i = 1, 2, \cdots$ and $v_n \to \infty$ as $n \to \infty$. By the definitions of $\{u_{ij}\}$ and $\{v_j\}$, we have $1 \leqq v_j \leqq v_{j+1}$ for all $j = 1, 2, \cdots$ and

$$\sum_{j=1}^{\infty} v_j a_j b_j^{-1} = \sum_{j=1}^{l_1-1} v_j a_j b_j^{-1} + \sum_{j=l_1}^{\infty} v_j a_j b_j^{-1} = \sum_{j=1}^{l_1-1} a_j b_j^{-1} + \sum_{j=l_1}^{\infty} \sum_{i=1}^{\infty} u_{ij} a_j b_j^{-1}$$

$$= \sum_{j=1}^{l_1-1} a_j b_j^{-1} + \sum_{i=1}^{\infty} \sum_{j=l_1}^{\infty} u_{ij} a_j b_j^{-1} \leqq \sum_{j=1}^{l_1-1} a_j b_j^{-1} + \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} u_{ij} a_j b_j^{-1} \leqq \sum_{j=1}^{l_1-1} a_j b_j^{-1} + \sum_{i=1}^{\infty} 2^{-i} < \infty.$$

Now, let $w_0 = 1$, $w_j = v_{2^{j-1}}$, $j = 1, 2, \cdots$, then we have:

(1) $1 \leqq w_j \leqq w_{j+1}$ for all $j = 0, 1, 2, \cdots$ and $w_j \to \infty$ as $j \to \infty$,

(2) $\sum_{n=0}^{\infty} w_n (\sum_{2^{n-1} < j \leqq 2^n} a_j b_j^{-1}) < \infty$.

Let $c_n^* = \min\{e_n, w_n\}$ for all $n = 0, 1, 2, \cdots$. Since $e_n \leqq e_{n+1}$, $w_n \leqq w_{n+1}$ for all $n = 0, 1, 2, \cdots$ and $e_n \to \infty$, $w_n \to \infty$ as $n \to \infty$, we have:

(1*) $1 \leqq c_n^* \leqq c_{n+1}^*$ for all $n = 0, 1, 2, \cdots$ and $c_n^* \to \infty$ as $n \to \infty$,

(2*) $c_n^* d_n \to 0$ as $n \to \infty$,

(3*) $\sum_{n=0}^{\infty} c_n^* (\sum_{2^{n-1} < j \leqq 2^n} a_j b_j^{-1}) < \infty$.

Now, we inductively define $\{c_n\}$ by letting $c_0 = 1$ and letting $c_n = \min\{c_n^*, c_{n-1} + 1\}$ for all $n = 1, 2, \cdots$. Then, we have:

(i) $c_n \leqq c_n^*$ for all $n = 0, 1, 2, \cdots$,

(ii) $1 \leqq c_n \leqq c_{n+1} \leqq c_n + 1$ for all $n = 0, 1, 2, \cdots$ and $c_n \to \infty$ as $n \to \infty$,

(iii) $c_n d_n = c_n b_{2^n}^{-1} \sum_{j=1}^{2^n} a_j \to 0$ as $n \to \infty$,

(iv) $\sum_{n=0}^{\infty} c_n (\sum_{2^{n-1} < j \leqq 2^n} a_j b_j^{-1}) < \infty$.

All, except the statement $c_n \to \infty$ as $n \to \infty$, are obvious; we prove "$c_n \to \infty$ as $n \to \infty$" as follows. First, suppose that there is a sequence $\{N_j\}$ of positive integers such that $C_{N_j} = c_{N_j}^*$ for all $j = 1, 2, \cdots$ and $N_j \to \infty$ as $j \to \infty$, then, by the fact $c_n \leqq c_{n+1}$ for all $n = 0, 1, 2, \cdots$, $c_n \to \infty$ as $n \to \infty$. Next, if there is a positive integer N such that $c_n \neq c_n^*$ for all $n > N$, then $c_{N+m} = c_N + m$ for all $m = 1, 2, \cdots$, hence $c_n \to \infty$ as $n \to \infty$. The proof of Lemma 2.5, is now complete.

Lemma 2.6. *Let $\sigma = \gamma_1 \times \gamma_2 \times \cdots$ be an independent strategy on H, $\{Y_n\}$ be a sequence of coordinate mappings on H, $S_0 = 0$, and $S_n = \sum_{j=1}^{n} Y_j$ for all $n = 1, 2, \cdots$. If $\varepsilon > 0, 0 < \delta < 1$, M, N are two integers such that $0 \leqq M < N < \infty$, and*

$$\max_{M \leqq n < N} \sigma([h \mid |S_N(h) - S_n(h)| > \varepsilon]) \leqq \delta,$$

then

$$\sigma([h \mid \max_{M < n \leqq N} |S_n(h) - S_M(h)| > 2\varepsilon]) \leqq \frac{1}{1-\delta} \sigma([h \mid |S_N(h) - S_M(h)| > \varepsilon]).$$

Proof. This result does not seem to have been stated before. The proof is essentially the same as the one in the conventional theory of probability (see [1]) and we omit it.

3. Borel Cantelli lemmas

In this section, we will state some finitely additive versions of Borel Cantelli lemmas. Lemma 3.2 and Lemma 2.5 in Section 2 are central results for proving the strong law of large numbers.

Let $\{K_n\}$ be a sequence of clopen subsets of H, $\{\tau_n\}$ be a strictly increasing sequence of stop rules on H such that K_n is determined by τ_n for all $n = 1, 2, \cdots$. From these two sequences $\{K_n\}$ and $\{\tau_n\}$, we define, for each positive integer n and each element $h = (x_1, x_2, \cdots)$ in H, a partial history $q_n(h) = p_{\tau_n}(h) = (x_1, x_2, \cdots, x_{\tau_n}(h))$ and a clopen subset $K_{n+1}q_n(h)$. The following is an important lemma for the σ-measure of countable intersections of clopen subsets of H.

LEMMA 3.1. *Let $\{K_n\}$, $\{\tau_n\}$, $\{q_n(h) \mid h \in H\}$, and $\{K_{n+1}q_n(h) \mid h \in H\}$ be as defined above. Let $g_1(h) = \chi_{K_1}(h)$ (the indicator function of the set K_1) and $g_n(h) = \sigma[q_{n-1}(h)](K_n q_{n-1}(h))$ for all $n = 2, 3, \cdots$. Suppose that $\{\alpha_n\}$ is a sequence of non-negative real numbers such that $0 \leqq \alpha_n \leqq 1$ for all $n = 1, 2, \cdots$. Let $K_1 \neq \emptyset$, $K_{n+1}q_n(h) \neq \emptyset$ for all $n \geqq 1$ and each h in $\bigcap_{i=1}^n K_i$, and let*

$$\sigma[q_{n-1}(h')]\left(\left[\bigcap_{i=1}^{n} K_i - \{h \mid g_{n+1}(h) \geqq \alpha_{n+1}\}\right] q_{n-1}(h')\right) = 0$$

for each $n = 1, 2, \cdots$, each h' in $\bigcap_{i=1}^{n-1} K_i$. Then we have $\sigma(\bigcap_{i=1}^\infty K_i) \geqq \sigma(K_1)\Pi_{j=2}^\infty \alpha_j$. If, in addition, $\sigma(K_1) \geqq \alpha_1$, then $\sigma(\bigcap_{i=1}^\infty K_i) \geqq \Pi_{j=1}^\infty \alpha_j$.

PROOF. Lemma 3.1 is a generalization of theorem 6.1 of [7], the proof is lengthy and is given in [2], and we omit it.

LEMMA 3.2. *Let $\{K_n\}$, $\{\tau_n\}$, $\{q_n(h) \mid h \in H\}$, $\{K_{n+1}q_n(h) \mid h \in H\}$, and $\{\alpha_n\}$ be as defined above. If*

(i) *there is a strictly increasing sequence $\{n_j\}$ of positive integers such that $\sigma(K_{n_j}^c) \to 1$ as $j \to \infty$,*

(ii) *for each $j = 1, 2, \cdots, m = 0, 1, 2, \cdots$, and each h in $\bigcap_{l=0}^m K_{n_j+l}^c$,*

$$\sigma[q_{n_j+m}(h)]\,(K^c_{n_j+m+1}\, q_{n_j+m}(h)) \geqq 1 - \alpha_{n_j+m+1}$$

and

$$\prod_{m=0}^{\infty} (1 - \alpha_{n_j+m+1}) \to 1 \quad \text{as} \quad j \to \infty,$$

then $\sigma([K_n \text{ i.o.}]) = \sigma(\bigcap_{m=1}^\infty \bigcup_{n=m}^\infty K_n) = 0$.

PROOF. Notice that $[K_n \text{ i.o.}] \subseteq \bigcup_{l=n_j}^\infty K_l$ for all $j = 1, 2, \cdots$. Hence

$$\sigma([K_n \text{ i.o.}]) \leqq \lim_{j\to\infty} \sigma\left(\bigcup_{l=n_j}^{\infty} K_l\right).$$

By Lemma 3.1,

$$\sigma\left(\bigcap_{l=n_j}^{\infty} K_l^c\right) \geqq \sigma(K_{n_j}^c) \prod_{m=0}^{\infty} (1-\alpha_{n_j+m+1}).$$

Hence

$$\lim_{j\to\infty} \sigma\left(\bigcap_{l=n_j}^{\infty} K_l^c\right) \geqq \lim_{j\to\infty}\left\{\sigma(K_{n_j}^c) \prod_{m=0}^{\infty} (1-\alpha_{n_j+m+1})\right\} = 1,$$

i.e.,

$$\lim_{j\to\infty} \sigma\left(\bigcup_{j\to\infty}^{\infty} K_l\right) \leqq 1 - \lim_{j\to\infty}\left\{\sigma(K_{n_j}^c) \prod_{m=0}^{\infty} (1-\alpha_{n_l+m+1})\right\} = 1-1=0.$$

Therefore, $\sigma([K_n \text{ i.o.}]) = 0$.

COROLLARY 3.1. *Let $\sigma = \gamma_1 \times \gamma_2 \times \cdots$ be an independent strategy on H, $\{N_j\}$ be a sequence of positive integers, $r_1 = 0$, and $r_n = \sum_{j=1}^{n-1} N_j$ for all $n = 2, 3, \cdots$. Suppose that A_n is a subset of X^{N_n}, $K_n = X^{r_n} \times A_n \times H$ for all $n = 1, 2, \cdots$, and $\sum_{n=1}^{\infty} \sigma(K_n) < \infty$. Then $\sigma([K_n \text{ i.o.}]) = 0$.*

LEMMA 3.3. *Let $\{K_n\}$, $\{q_n(h) \mid h \in H\}$, $\{K_{n+1} q_n(h) \mid h \in H\}$, and $\{\alpha_n\}$ be as defined in Lemma 3.2. Suppose that, for each $n = 1, 2, \cdots$, for all h in H, $\sigma[q_n(h)](K_{n+1} q_n(h)) \geqq \alpha_{n+1}$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then $\sigma([K_n \text{ i.o.}]) = 1$.*

PROOF. See pp. 36–37 of [7].

COROLLARY 3.2. *Let σ and $\{K_n\}$ be as in Corollary 3.1 and let $\sum_{n=1}^{\infty} \sigma(K_n) = \infty$. Then $\sigma([K_n \text{ i.o.}]) = 1$.*

4. Strong laws of large numbers

Now, we are in the position to prove the strong law of large numbers in our finitely additive setting.

THEOREM 4.1. *Let $\sigma = \gamma_1 \times \gamma_2 \times \cdots$ be an independent strategy on H, $\{Y_n\}$ be a sequence of coordinate mappings on H. If $\sigma(Y_n) = 0$ for all $n = 1, 2, \cdots$, and, for some constant $r \geqq 1$, $\sum_{n=1}^{\infty} \{\sigma(|Y_n|^{2r})/n^{1+r}\} < \infty$, then the set*

$$A = \left[h \mid \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} Y_j(h) = 0\right]$$

has σ-measure 1.

Proof. Since $\{Y_n\}$ is a sequence of coordinate mappings on H, A is a Borel subset of H. Since $\mathscr{A}(\sigma)$ contains all Borel subsets of H (see section 5 of [7]), $\sigma(A)$ is well defined.

First, we choose, as is possible by Lemma 2.5, a sequence $\{c_n\}$ of positive real numbers such that:

(i) $c_n(2^n)^{-(1+r)}\Sigma_{j=1}^{2^n}\sigma(|Y_j|^{2r})\to 0$ as $n\to\infty$,

(ii) $1\leqq c_n\leqq c_{n+1}\leqq c_n+1$ for all $n=0,1,2,\cdots$ and $c_n\to\infty$ as $n\to\infty$,

(iii) $\Sigma_{n=0}^{\infty}c_n\{\Sigma_{2^{n-1}<j\leqq 2^n}j^{-(1+r)}\sigma(|Y_j|^{2r})\}<\infty$.

Now, we prove Theorem 4.1 in the following three steps (let $S_0=0$, $S_n=\Sigma_{j=1}^{n}Y_j$, $S_{m,n}=S_n-S_m$ for all $0\leqq m<n<\infty$):

Step 1. $\sigma([h\mid\lim_{n\to\infty}\{(2^n)^{-1}S_{2^n}(h)\}=0])=1$.
To see this, let

$$K_n=\left[h\,\middle|\,\frac{1}{2^n}|S_{2^n}(h)|>c_n^{-1/4r}\right],\qquad n=1,2,\cdots.$$

By Markov's inequality, we have

$$\sigma(K_n)\leqq c_n^{1/2}\sigma(\{(2^n)^{-2r}|S_{2^n}|^{2r}\})=c_n^{1/2}(2^n)^{-2r}\sigma(|S_{2^n}|^{2r}).$$

By Lemma 2.4, $\sigma(|S_{2^n}|^{2r})\leqq A(2^n)^{r-1}\Sigma_{j=1}^{2^n}\sigma(|Y_j|^{2r})$, where A is a positive constant. Hence

$$\sigma(K_n)\leqq Ac_n(2^n)^{-(1+r)}\sum_{1\leqq j\leqq 2^n}\sigma(|Y_j|^{2r})\quad(\text{since } c_n\geqq 1).$$

By (i) above, $\sigma(K_n)\to 0$ as $n\to\infty$, i.e., $\sigma(K_n^c)\to 1$ as $n\to\infty$. Now, for each $n=1,2,\cdots$, we define a stop rule $\tau_n=2^n$ on H and, for each $n=1,2,\cdots$, each h in H, we define a subset $A_{n+1}(h)$ of H by

$$A_{n+1}(h)=\{h'\mid\,|S_{2^n,2^{n+1}}(q_n(h)h')|\leqq(c_{n+1}^{-1/4r}2^{n+1}-c_n^{-1/4r}2^n)\}.$$

It is easy to check that, for each $n=1,2,\cdots$, $A_{n+1}(h)\subseteq K_{n+1}^c q_n(h)$ if h is in $\bigcap_{l=1}^n K_l^c$. Hence, for each $n=1,2,\cdots$,

$$\sigma[q_n(h)](K_{n+1}^c q_n(h))\geqq\sigma[q_n(h)](A_{n+1}(h))$$
$$\geqq 1-(c_{n+1}^{-1/4r}2^{n+1}-c_n^{-1/4r}2^n)^{-2r}\sigma[q_n(h)](|S_{2^n,2^{n+1}}q_n(h)|^{2r})$$

if h is in $\bigcap_{l=1}^n K_l^c$. By Lemma 2.2, the last expression is equal to

$$1-(c_{n+1}^{-1/4r}2^{n+1}-c_n^{-1/4r}2^n)^{-2r}\sigma(|S_{2^n,2^{n+1}}|^{2r}).$$

Hence, if h is in $\bigcap_{l=1}^{n} K_l^c$,

$$\sigma[q_n(h)](K_{n+1}^c q_n(h)) \geqq 1-(2^{n+1})^{-2r}\left(c_{n+1}^{-1/4r}-\frac{1}{2}c_n^{-1/4r}\right)^{-2r} A(2^n)^{r-1}\sum_{2^n<j\leqq 2^{n+1}}\sigma(|Y_j|^{2r})$$

$$=1-(2^n)^{-(1+r)}2^{-2r}\left(c_n^{1/4r}-\frac{1}{2}c_{n+1}^{1/4r}\right)^{-2r} A(c_nc_{n+1})^{1/2}\sum_{2^n<j\leqq 2^{n+1}}\sigma(|Y_j|^{2r}).$$

Since $1\leqq c_n\leqq c_{n+1}\leqq c_n+1$ for all $n=0,1,2,\cdots$ and $c_n\to\infty$ as $n\to\infty$, there is a positive integer N such that $c_n^{1/4r}-\frac{1}{2}c_{n+1}^{1/4r}\geqq\frac{1}{2}$ for all $n\geqq N$. Therefore, if $n\geqq N$ and h is in $\bigcap_{l=1}^{n} K_l^c$,

$$\sigma[q_n(h)](K_{n+1}^c q_n(h)) \geqq 1-(2^n)^{-(1+r)}Ac_{n+1}\sum_{2^n<j\leqq 2^{n+1}}\sigma(|Y_j|^{2r}).$$

By (iii) above,

$$\sum_{n=N}^{\infty}(2^{n+1})^{-(1+r)}c_{n+1}\sum_{2^n<j\leqq 2^{n+1}}\sigma(|Y_j|^{2r})<\infty$$

and it is equivalent to

$$\prod_{l=n}^{\infty}\{1-(2^l)^{-(1+r)}Ac_{l+1}\sum_{2^l<j\leqq 2^{l+1}}\sigma(|Y_j|^{2r})\}\to 1 \quad\text{as}\quad n\to\infty.$$

By Lemma 3.2, $\sigma([K_n \text{ i.o.}])=0$, i.e., $\sigma([K_n \text{ i.o.}]^c)=1$. But the set $[h \mid \lim_{n\to\infty}1/2^n S_{2^n}(h)=0]$ contains the set $[K_n \text{ i.o.}]^c$, hence

$$\sigma([h \mid \lim_{n\to\infty}\frac{1}{2^n}S_{2^n}(h)=0])=1.$$

Step 2. $\sigma([h \mid \lim_{n\to\infty}1/2^n \max_{2^n<m\leqq 2^{n+1}}|S_m(h)-S_{2^n}(h)|=0])=1$.
To see this, let $D_n=\max_{2^n<m\leqq 2^{n+1}}|S_{2^n,m}|$, $n=1,2,\cdots$. Since

$$\sum_{n=0}^{\infty}c_n\sum_{2^{n-1}<j\leqq 2^n}\{\sigma(|Y_j|^{2r})/j^{1+r}\}<\infty,$$

there is a positive integer N_1 such that

$$A\sum_{j=n}^{\infty}c_j\sum_{2^{j-1}<l\leqq 2^j}\{\sigma(|Y_l|^{2r})/l^{1+r}\}\leqq 2^{-(3+r)}$$

for all $n\geqq N_1$, where A is the positive constant in the proof of Step 1. Now if $n\geqq N_1$, then

$$\sigma([\,|S_{2^{n+1}} - S_m| > 2^n c_n^{-1/2r}]) \leqq (2^n)^{-2r} c_n A (2^{n+1} - m)^{r-1} \sum_{m<j\leqq 2^{n+1}} \sigma(|Y_j|^{2r})$$

$$\leqq (2^n)^{-(1+r)} c_n A \sum_{2^n<j\leqq 2^{n+1}} \sigma(|Y_j|^{2r})$$

for all m such that $2^n \leqq m < 2^{n+1}$. Hence, if $n \geqq N_1$,

$$\max_{2^n \leqq m < 2^{n+1}} \sigma([\,|S_{2^{n+1}} - S_m| > 2^n c_n^{-1/2r}]) \leqq (2^n)^{-(1+r)} c_n A \sum_{2^n<j\leqq 2^{n+1}} \sigma(|Y_j|^{2r}) \leqq \frac{1}{4}.$$

By Lemma 2.6, if $n \geqq N_1$,

$$\sigma([D_n > 2 \cdot 2^n c_n^{-1/2r}]) \leqq \frac{1}{1-\frac{1}{4}} \sigma([\,|S_{2^n, 2^{n+1}}| > 2^n c_n^{-1/2r}])$$

$$\leqq \frac{4A}{3} (2^n)^{-(1+r)} c_n \sum_{2^n<j\leqq 2^{n+1}} \sigma(|Y_j|^{2r}) \leqq \frac{2^{3+r}}{3} A c_n \left\{ \sum_{2^n<j\leqq 2^{n+1}} \sigma(|Y_j|^{2r})/j^{1+r} \right\}.$$

Set $L_n = [h \mid D_n(h) > 2 \cdot 2^n c_n^{-1/2r}]$. Then, if $n \geqq N_1$,

$$\sigma(L_n) \leqq \frac{2^{3+r}}{3} A c_n \sum_{2^n<j\leqq 2^{n+1}} \{\sigma(|Y_j|^{2r})/j^{1+r}\}.$$

Hence $\sum_{n=1}^{\infty} \sigma(L_n) < \infty$. By Corollary 3.1, $\sigma([L_n \text{ i.o.}]^c) = 1$ and notice that the set $[h \mid \lim_{n\to\infty} 1/2^n D_n(h) = 0]$ contains the set $[L_n \text{ i.o.}]^c$. Therefore

$$\sigma\left(\left[h \mid \lim_{n\to\infty} \frac{1}{2^n} \max_{2^n<m\leqq 2^{n+1}} |S_m(h) - S_{2^n}(h)| = 0\right]\right) = 1.$$

Step 3. $\sigma([h \mid \lim_{n\to\infty} \frac{1}{n} S_n(h) = 0]) = 1$.
To see this, let us define $m(n)$ as the integer such that $2^{m(n)} \leqq n < 2^{m(n)+1}$, $n = 2, 3, \cdots$. It is easy to check that

$$\left[h \mid \lim_{n\to\infty} \frac{1}{n} S_n(h) = 0\right] \supseteq \left[h \mid \lim_{n\to\infty} \frac{1}{2^{m(n)}} D_{m(n)}(h) = 0\right] \cap \left[k \mid \lim_{n\to\infty} \frac{1}{2^{m(n)}} S_{2^{m(n)}}(h) = 0\right]$$

and therefore,

$$\sigma\left(\left[h \mid \lim_{n\to\infty} \frac{1}{n} S_n(h) = 0\right]\right) = 1.$$

The proof of Theorem 4.1 is now complete.

COROLLARY 4.1. *Suppose that $\sigma = \gamma_1 \times \gamma_2 \times \cdots$ is an independent strategy on H, $\{Y_n\}$ is a sequence of coordinate mappings on H such that $\sigma(Y_n) = 0$ for all $n = 1, 2 \cdots$ and $\Sigma_{n=1}^{\infty}\{\sigma(|Y_n|^2)/n^2\} < \infty$. Then*

$$\sigma\left(\left[h \,\Big|\, \lim_{n\to\infty} \frac{1}{n} S_n(h) = 0\right]\right) = 1.$$

THEOREM 4.2. *Suppose that $\sigma = \gamma_1 \times \gamma_2 \times \cdots$ is an independent strategy on H, $\{a_n\}$ is a nondecreasing sequence of positive real numbers such that $a_n \to \infty$ as $n \to \infty$, and $\{Y_n\}$ is a sequence of coordinate mappings on H such that $\sigma(Y_n) = 0$ for all $n = 1, 2, \cdots$ and $\Sigma_{n=1}^{\infty}\{\sigma(Y_n^2)/a_n^2\} < \infty$. Then*

$$\sigma\left(\left[h \,\Big|\, \lim_{n\to\infty} \frac{1}{a_n} \sum_{j=1}^{n} Y_j(h) = 0\right]\right) = 1.$$

PROOF. For each $n = 1, 2, \cdots$, let $Y_n^* = Y_n / a_n$. Then $\sigma(Y_n^*) = 0$ and $\Sigma_{n=1}^{\infty} \sigma(Y_n^{*2}) < \infty$. Let $N_0 = 0$ and

$$N_j = \max\left\{N_{j-1} + 1, \inf\left\{n \,\Big|\, \sum_{i=n}^{\infty} \sigma(Y_i^{*2}) < (1+j)^{-6}\right\}\right\} \quad \text{for all} \quad j = 1, 2, \cdots.$$

Since $\Sigma_{n=1}^{\infty} \sigma(Y_n^{*2}) < \infty$, $1 \leqq N_1 < N_2 < \cdots < \infty$ and $N_j \to \infty$ as $j \to \infty$. For each $j = 1, 2, \cdots$, let

$$D_j = \max_{N_j < l \leqq N_{j+1}} \left| \sum_{N_j < n \leqq l} Y_n^* \right|, \quad L_j = [h \mid D_j(h) > 2(1+j)^{-2}].$$

Then, for $N_j < l \leqq N_{j+1}$,

$$\sigma\left(\left[\left| \sum_{l \leqq n \leqq N_{j+1}} Y_n^* \right| > (1+j)^{-2}\right]\right) \leqq (1+j)^4 \sigma\left(\left| \sum_{l \leqq n \leqq N_{j+1}} Y_n^* \right|^2\right)$$

$$= (1+j)^4 \sum_{l \leqq n \leqq N_{j+1}} \sigma(|Y_n^*|^2)$$

$$\leqq (1+j)^4 \sum_{n=N_j+1}^{\infty} \sigma(|Y_n^*|^2) \leqq (1+j)^{-2}$$

for all $j = 1, 2, \cdots$. By Lemma 2.6,

$$\sigma(L_j) \leqq \frac{1}{1-(1+j)^{-2}}\sigma\left(\left[\left|\sum_{N_j<n\leqq N_{j+1}} Y_n^*\right| > (1+j)^{-2}\right]\right) \leqq \frac{(1+j)^{-2}}{1-(1+j)^{-2}} < 2(1+j)^{-2}$$

for all $j = 1, 2, \cdots$. Hence $\Sigma_{j=1}^{\infty}\sigma(L_j) < \infty$. By Corollary 3.1, $\sigma([L_j \text{ i.o.}]^c) = 1$. Notice that the set $[h \mid \lim_{n\to\infty}\Sigma_{j=1}^{n} Y_j^*(h)$ exists and is finite$]$ contains the set $[L_j \text{ i.o.}]^c$. Hence $\sigma([h \mid \lim_{n\to\infty}\Sigma_{j=1}^{n} Y_n^*(h)$ exists and is finite$]) = 1$. By Kronecker's lemma, we have

$$\sigma\left(\left[h \mid \lim_{n\to\infty}\frac{1}{a_n}\sum_{j=1}^{n} a_j Y_j^*(h) = 0\right]\right) = \sigma\left(\left[h \mid \lim_{n\to\infty}\frac{1}{a_n}\sum_{j=1}^{n} Y_j(h) = 0\right]\right) = 1.$$

The proof of Theorem 4.2 is now complete.

REMARK. In [2], Theorem 4.2 is proved by using a convergence theorem of Lévy in this finitely additive setting.

COROLLARY 4.2. *Let $\sigma = \gamma_1 \times \gamma_2 \times \cdots$ be an independent strategy on H and $\{Y_n\}$ be a sequence of coordinate mappings on H such that $\sigma(Y_n) = 0$ and $\sigma(Y_n^2) < \infty$ for all $n \geqq 1$. Suppose that $\sigma(Y_n^2) = O(n^\theta)$ as $n \to \infty$ for some constant $\theta \geqq -1$. Then*

$$\sigma\left(\left[h \mid \lim_{n\to\infty}\frac{1}{n^\alpha}\sum_{j=1}^{n} Y_j(h) = 0\right]\right) = 1$$

for any constant $\alpha > (1+\theta)/2$.

COROLLARY 4.3. *Let $\sigma = \gamma_1 \times \gamma_2 \times \cdots$ be an independent strategy on H and $\{Y_n\}$ is a sequence of coordinate mappings on H such that $\sigma(Y_n) = 0$, $\sigma(Y_n^2) \leqq M < \infty$ for all $n = 1, 2, \cdots$. Then*

$$\sigma\left(\left[h \mid \lim_{n\to\infty}\frac{1}{n^\alpha}\sum_{j=1}^{n} Y_j(h) = 0\right]\right) = 1$$

for any $\alpha > \frac{1}{2}$.

The next theorem is the finitely additive version of Kolmogorov's strong law of large numbers for independent, identically distributed strategies and sequences of identical, coordinate mappings. The theorem is a generalization of Kolmogorov's strong law of large numbers in a coordinate representation process. Before proving the theorem, we need a lemma.

LEMMA 4.1. *Let $\sigma = \gamma \times \gamma \times \cdots$ be an independent, identically distributed* *strategy on H, $\{Y_n\}$ be a sequence of identical, coordinate mappings on H, and f be a real-valued function defined on X such that $f(x_1) = Y_1(h)$ whenever $h = (x_1, x_2, \cdots)$ is in H. Then Y_1, $Y_2, \cdots$ are σ-integrable if and only if f is γ-integrable. Furthermore, $\gamma(f) = \sigma(Y_1) = \sigma(Y_2) = \cdots$ whenever these integrals exists.*

Proof. Since $Y_n(h) = f(x_n)$ if $h = (x_1, x_2, \cdots, x_n, \cdots)$ is in H, $n = 1, 2, \cdots$, the statement follows from Lemma 2.1.

Theorem 4.3. *Let $\sigma = \gamma \times \gamma \times \cdots$ be an independent, identically distributed strategy on H, $\{Y_n\}$ be a sequence of identical, coordinate mappings on H. Then*

$$\sigma\left(\left[h \mid \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} Y_j(h) = u\right]\right) = 1$$

if and only if Y_1, $Y_2, \cdots$ are σ-integrable and $\sigma(Y_1) = \sigma(Y_2) = \cdots u$, where u is a real number.

Proof. Let us define a real-valued function f on X by $f(x_1) = Y_1(h)$ if $h = (x_1, x_2, \cdots)$. By Lemma 4.1, Y_1, $Y_2, \cdots$ are σ-integrable if and only if f is γ-integrable.

First, we assume that f is γ-integrable and $\gamma(f) = u$. Whithout loss of generality, we assume that $u = 0$. For each $n = 1, 2, \cdots$, let $Y_n^*(h) = Y_n(h)$ if $|Y_n(h)| \geqq n$, $Y_n^*(h) = 0$ if $|Y_n(h)| > n$ and $f_n^*(x_n) = Y_n^*(h)$ if $h = (x_1, x_2, \cdots, x_n \cdots)$ in H. Notice that Y_n^* is σ-integrable, f_n^* is γ-integrable and $\sigma(Y_n^*) = \gamma(f_n^*)$.

Since $|f_n^*| \leqq |f|$ for all $n = 1, 2, \cdots$, $f_n^* \to f$ in γ-measure as $n \to \infty$, by the dominated convergence theorem for the finitely additive setting (see pp. 124–125 of [6]), we have $\gamma(f_n^*) \to \gamma(f) = 0$ as $n \to \infty$.

Set $Z_n = Y_n^* - a_n$, where $a_n = \gamma(f_n^*) = \sigma(Y_n^*)$, $n = 1, 2, \cdots$. Then

$$\sum_{n=1}^{\infty} \{\sigma(Z_n^2)/n^2\} \leqq \sum_{n=1}^{\infty} \{\sigma(Y_n^{*2})/n^2\} = \sum_{n=1}^{\infty} \frac{1}{n^2} \int Y_n^2 \chi_{[|Y_n| \leqq n]}\, d\sigma.$$

Notice that

$$\begin{aligned}\int_n Y_n^2 \chi_{[|Y_n| \leqq n]}\, d\sigma = \int f_n^{*2}\, d\gamma &= \sum_{j=0}^{n-1} \int_{[j < |f^*| \leqq j+1]} f_n^{*2}\, d\gamma \\ &\leqq \sum_{j=0}^{n-1} (1+j)^2 \gamma(\{x \mid j < |f_m^*(x)| \leqq j+1\}) \\ &= \sum_{j=0}^{n-1} (1+j)^2 \gamma(\{x \mid j < |f(x)| \leqq 1+j\}).\end{aligned}$$

Hence

$$\sum_{n=1}^{\infty}\{\sigma(|Y_n^*|^2)/n^2\} \leqq \sum_{n=1}^{\infty}\sum_{j=0}^{n-1}\frac{(1+j)^2}{n^2}\gamma(\{x \mid j < |f(x)| \leqq 1+j\})$$

$$= \sum_{j=1}^{\infty}\sum_{n=j}^{\infty}\frac{j^2}{n^2}\gamma(\{x \mid j-1 < |f(x)| \leqq j\}) \leqq \sum_{j=1}^{\infty}\frac{2j^2}{j}\gamma(\{x \mid j-1 < |f(x)| \leqq j\})$$

$$= \sum_{j=1}^{\infty}2(j-1)\gamma(\{x \mid j-1 < |f(x)| \leqq j\}) + 2\sum_{j=1}^{\infty}\gamma(\{x \mid j-1 < |f(x)| \leqq j\})$$

$$\leqq 2\gamma(|f|)+1 < \infty.$$

(The last step is implied by theorem 1.2 in [2]). By Theorem 4.1, we have

$$\sigma\left(\left[h \mid \lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n} Z_j(h) = 0\right]\right) = 1.$$

Since $a_n \to 0$ as $n \to \infty$, $\lim_{n\to\infty} 1/n \sum_{j=1}^{n} a_j = 0$. Hence

$$\sigma\left(\left[h \mid \lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n} Y_j^*(h) = 0\right]\right) = 1.$$

For each $n = 1, 2, \cdots$, set $K_n = [Y_n^* \neq Y_n]$,

$$K_n^* = [h \mid h = (x_1, x_2, \cdots, x_n, \cdots), |f(x_n)| > n].$$

It is easy to see that

$$\sigma(K_n) = \sigma(K_n^*) = \gamma(\{x \mid |f(x)| > n\}).$$

By theorem 1.2 of [2], $\sum_{n=1}^{\infty}\gamma(\{x \mid |f(x)| > n\}) < \infty$ if and only if $\gamma(|f|) < \infty$. Therefore $\sum_{n=1}^{\infty}\sigma(K_n) < \infty$ and, by Corollary 3.1, $\sigma([K_n \text{ i.o.}]^c) = 1$. Since

$$[K_n \text{ i.o.}]^c \cap \left[h \mid \lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n} Y_j^*(h) = 0\right] \subseteq \left[h \mid \lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n} Y_j(h) = 0\right],$$

$$\sigma\left(\left[h \mid \lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n} Y_j(h) = 0\right]\right) = 1.$$

Next, suppose that

$$\sigma\left(\left[h \mid \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} Y_j(h) = 0\right]\right) = 1,$$

then

$$\sigma\left(\left[h \mid \lim_{n\to\infty} \frac{Y_n(h)}{n} = 0\right]\right) = 1.$$

For each $n = 1, 2, \cdots$, set $L_n = [h \mid |Y_n(h)| > n]$. If $\sum_{n=1}^{\infty} \sigma(L_n) = \infty$, then, by Corollary 3.2, $\sigma([L_n \text{ i.o.}]) = 1$. Hence, it is necessary that $\sum_{n=1}^{\infty} \sigma(L_n) < \infty$. Let $L_n^* = [x \mid |f(x)| > n]$ for all $n = 1, 2, \cdots$. Then $\sigma(L_n) = \gamma(L_n^*)$ for all $n = 1, 2, \cdots$ and

$$\sum_{n=1}^{\infty} \sigma(L_n) = \sum_{n=1}^{\infty} \gamma(L_n^*) < \infty.$$

By theorem 1.2 of [2], f is γ-integrable, hence, by Lemma 4.1, $Y_1, Y_2, \cdots$ are σ-integrable. By the first part of the proof above, we should have $\gamma(f) = \sigma(Y_1) = \sigma(Y_2) = \cdots = 0$. The proof of Theorem 4.3 is now complete.

Remark. In [2], it was shown that if $\gamma(f^+) = \infty$, $\gamma(f^-) < \infty$, then

$$\sigma\left(\left[h \mid \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} Y_j(h) = \infty\right]\right) = 1.$$

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MIAMI
CORAL GABLES, FLORIDA 33124, USA